Lecture 34– The Newton-Raphson Method

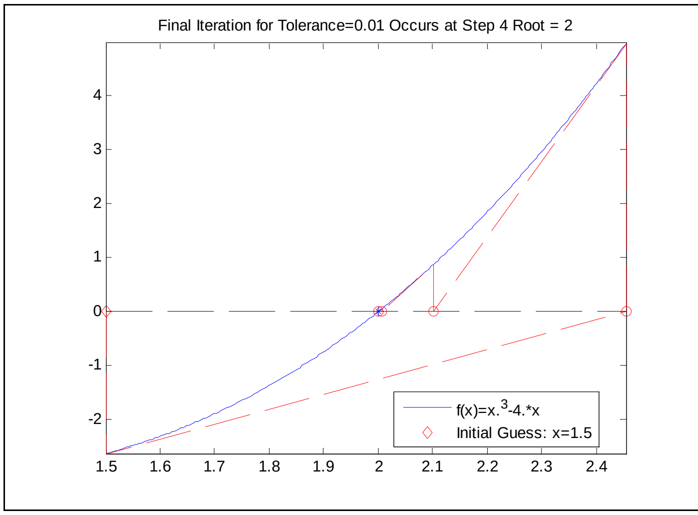
- Used to find a roots of a complicated functions
  - Using concepts of CALC I, we can numerically evaluate roots.
  - The Newton-Raphson method is an iterative process used approximate roots of a function.
  - It is given by the formula:

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

- Consider the formula $x^3 - 4x = 0$. This relatively simple equation has roots at x=2, x=-2, x=0.
- If we don't know this in advance we can use the iterative equation to numerically obtain the roots:

$$x_{n+1} = x_n - \frac{x_n^3 - 4x_n}{3x_n^2 - 4}$$

- An initial guess must be provided and tolerance criteria (i.e. criteria that declares convergence of $|x_{n+1} - x_n| < \partial$). If I let $x_0 = 1.5$ and $\partial = .01$ I get the following results:

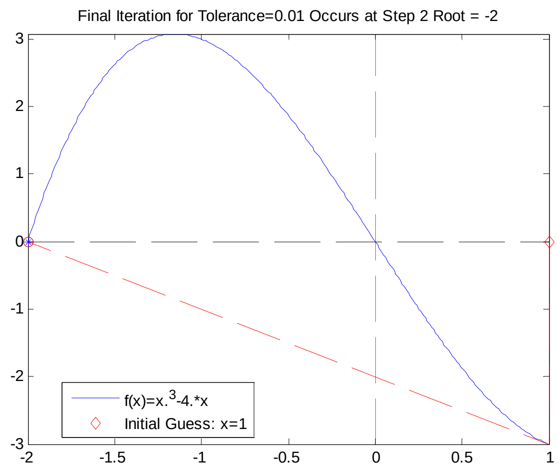


Step	x_n
0	1.5000
1	2.4545
2	2.1014
3	2.0069
4	2.0000

Note that the initial guess gave us the root *x=2.*

  - The cobweb plot at the right is generated by plotting the following for each iteration.
    - Plot $(x_n, 0)$ to $(x_n, f(x_n))$
    - Then plot $(x_n, f(x_n))$ to $(x_{n+1}, 0)$

- Now I change the initial guess: $x_0 = 1$. Note that a single iteration lands me another root *x=-1.*

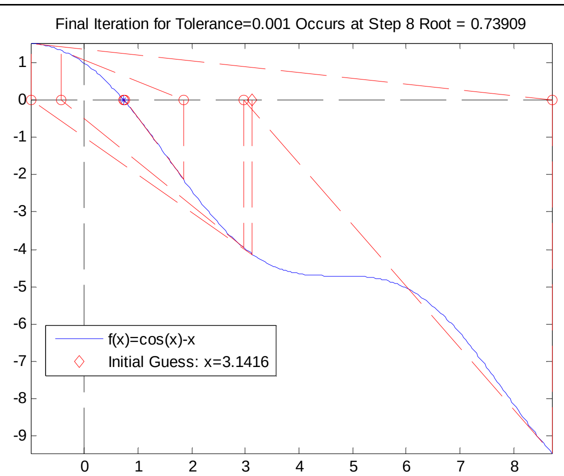


Step	x_n
0	1
1	-2

Note that the initial guess gave us the root *x=-2.* If I had guessed $x_0 = .85$ I would land on the root *x=0*

- There in lies the conundrum for the Newton-Raphson Method:
  - It will numerically estimate a root, but which root?
  - How many roots are there? You must know something about the equation to answer this question.

- Here is another interesting example: Find the solution to $\cos(x) = x$. To use Newton's method, we rewrite the equation $f(x) = \cos(x) - x$ and find the roots for $f(x)$. In the table below I use $x_0 = \pi$ and $\partial = .001$.



Step	x_n
0	3.1416
1	-1.0000
2	8.7162
3	2.9761
4	-0.4258
5	1.8512
6	0.7660
7	0.7392
8	0.7391

  - If I plot out f(x), I note that it has only one root, therefore I do not need to probe for more roots.

  - You can experiment more with the applet listed in on the webpage: http://www.math.umn.edu/~garrett/qy/Newton.html

- Who the Heck is Raphson?:

Joseph Raphson was an English mathematician known best for the Newton-Raphson method. Little is known about Raphson's life - even his exact birth and death years are unknown, though the mathematical historian Florian Cajori supplied the approximate dates 1648-1715. Raphson attended Jesus College in Cambridge and graduated with an M.A. in 1692. Raphson was made a Fellow of the Royal Society in 30 November 1689 after being proposed for membership by Edmund Halley.

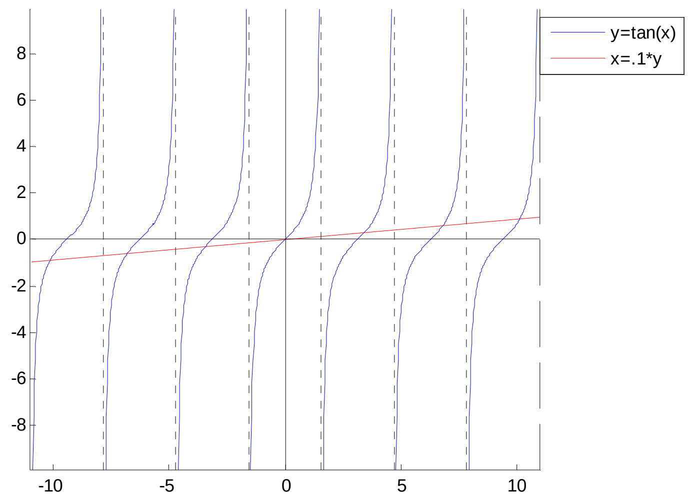
Raphson's most notable work was *Analysis Aequationum Universalis* which was published in 1690. It contained the Newton-Raphson method for approximating the roots of an equation. Isaac Newton developed the same formula in the *Method of Fluxions*. Although Newton's work was written in 1671, it was not published until 1736- nearly 50 years after Raphson's *Analysis* . Furthermore, Raphson's version of the method is simpler and therefore superior, and it is this version that is found in textbooks today.

Raphson was a staunch supporter of Newton's claims as the inventor of Calculus against Gottfried Leibniz's. In addition, Raphson translated Newton's Arithmetica Universalis into English. The two were not close friends however, as evidenced by Newton's inability to spell Raphson's name either correctly or consistently.

Citation: http://en.wikipedia.org/wiki/Joseph_Raphson

Homework

The equation $\tan(x) = .1x$ appears when finding eigenvalues that result from solving the heat equation in a bar whose ends have mixed boundary conditions (i.e. convection). It has in infinite number of solutions as depicted by the figure below.



Write a MATLAB program that uses the Newton-Raphson method to find the first three **positive** solutions. Hint: Use the plot above to provide an initial guess for x. Note: MATLAB does not automatically find derivatives … you must code these by hand.

Your homework should include:

- Your code
- The value of the three roots (use a tolerance of .001).
- The answer to these questions:
  - If you guess is to close to the vertical asymptote, Newton-Raphson Method becomes unstable and "blows-up". Why?
  - Where else does the routine become unstable?
- <u>You do not need to turn in cobweb plots.</u>